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THE LINDELÖF NUMBER GREATER THAN CONTINUUM IS u-INVARIANT

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Abstract. Two Tychonoff spaces X and Y are said to be l-equivalent (u-equivalent) if $C_p(X)$ and $C_p(Y)$ are linearly (uniformly) homeomorphic. N. V. Velichko proved that countable Lindelöf number is preserved by the relation of l-equivalence. A. Bouziad strengthened this result and proved that any Lindelöf number is preserved by the relation of l-equivalence. In this paper it has been proved that the Lindelöf number greater than continuum is preserved by the relation of u-equivalence.

Introduction. Our aim is to prove the following main result of the paper.

Theorem 0.1. *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic and the Lindelöf number of X or Y greater than continuum. Then $l(X) = l(Y)$.*

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For the proof we need some auxiliary concepts. In the first section, we consider set-valued mappings K and K_ε of the space X to Y generated by the uniform homeomorphism of the spaces $C_p(Y)$ and $C_p(X)$, and formulate their properties. In the second section, we prove the main result. Section 3 is devoted to the proof of the auxiliary results.

Terminology and notations. In notation and terminology we follow R. Engelking's book [2]. The spaces considered in this paper are taken to be Tychonoff spaces. The symbols X, Y are used only for topological spaces. $\mathbb{R}$ denotes the usual space of real numbers, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers. The symbol $\overline{k, m}$ denotes the set of all natural numbers n such that $k \leq n \leq m$, where $k, m \in \mathbb{N}$, $k \leq m$. $\mathbb{R}^X$ is a space of all real-valued functions on X, $C_p(X)$ is a space of all real-valued continuous functions on X equipped with the topology of pointwise convergence. $\operatorname{Fin}\mathcal{F}$ is a family of all finite subsets of a set $\mathcal{F}$.

The restriction of the mapping f to the subset A is denoted by $f|_A$. $f^{-1}(A)$ is a preimage of the set A under the mapping f. If A is an interval, then we shall use the symbol $f^{-1}A$ instead of $f^{-1}(A)$. $|A|$ denotes the cardinality of A, $\operatorname{Int} A$ denotes the interior of A. A subset A of X will be called functionally closed (functionally open) if $A = f^{-1}(1)$ ($A = f^{-1}(0, 1]$ respectively) for some continuous function $f\colon X \to [0, 1]$. We say that the set A is a G_δ-subset of X if A can be represented as the intersection of some countable family of open subsets of X.

The cardinal number assigned to the set of all positive integers is denoted by the symbol $\aleph_0$, and the cardinal number assigned to the set of all real numbers is denoted by c (continuum). The symbol τ denote infinite cardinal only. For any cardinal number τ symbol $\omega(\tau)$ denotes the initial ordinal number λ such that $|\lambda| = \tau$. The Lindelöf number $l(X)$ of a space X is the smallest infinite cardinal τ such that any open cover of X contains a subcover of cardinality at most τ.

For a set-valued mapping $p : X \to Y$ and sets $A \subset X$ and $B \subset Y$, the set $p(A) = \bigcup\{p(x)\colon x \in A\}$ is called the image of A under p, and the set $p^{-1}(B) = \{x \in X\colon p(x) \cap B \neq \varnothing\}$ is called the preimage of B under p. A set-valued mapping $p : X \to Y$ is called lower semicontinuous if for every open subset of Y its preimage under p is open in X, and p is called surjective if for every $y \in Y$ there exists an $x \in X$ such that $y \in p(x)$.

1. Set-valued mappings concerned with uniform homeomorphisms of function spaces and their properties.

Definition 1.1. *Let $h\colon C_p(Y) \to C_p(X)$ be a uniform homeomorphism. Fix $x \in X$, $\delta > 0$, and finite subset $K \subset Y$, and put*

$$a(x,K,\delta) = \sup\{|h(g')(x) - h(g'')(x)| : \\ g', g'' \in C_p(Y),\ |g'(y) - g''(y)| < \delta \textit{ for all } y \in K\}.$$

This notion was introduced by S. P. Gul'ko in [3]. Next, we define

$$a(x,K,0) = \sup\{|\,h(g')(x) - h(g'')(x)| : \\ g', g'' \in C_p(Y),\ g'(y) = g''(y) \text{ for all } y \in K\}.$$

(if the set K is empty, then the supremum is taken over all $g', g'' \in C_p(Y)$). It is obvious that if $0 \le \delta_1 \le \delta_2$, then $a(x,K,\delta_1) \le a(x,K,\delta_2)$, and if $K_1 \subset K_2 \subset Y$, then $a(x,K_2,\delta) \le a(x,K_2,\delta)$ for all $\delta \ge 0$. It was proved in [3] that for every $x \in X$ there exists a nonempty finite subset $K(x) \subset Y$ such that

1. $a(x,K(x),\delta) < \infty$ for any $\delta > 0$,
2. $a(x,K',\delta) = \infty$ for every proper subset K' of $K(x)$ and for any $\delta > 0$,
3. If $a(x,K,\delta) < \infty$ for some finite subset $K \subset Y$ and $\delta > 0$, then $K(x) \subset K$.

S. P. Gul'ko also proved that if $a(x,K,\delta_0) < \infty$ for some finite subset $K \subset Y$ and $\delta_0 > 0$, then $a(x,K,\delta) < \infty$ for all $\delta > 0$. We now prove that the set $K(x)$ has the following property, which is stronger than the property 2.

4. $a(x,K',0) = \infty$ for every proper subset K' of $K(x)$.

To prove this statement we need the following

Lemma 1.2. *If $a(x,K,0) < \infty$, then $a(x,K,\delta) < \infty$ for all $\delta > 0$.*

Proof. Fix $x \in X$ and finite subset $K \subset Y$ such that $a(x,K,0) < \infty$. We prove that the function $\delta \mapsto a(x,K,\delta)$ is continuous at the point 0. Let $\varepsilon > 0$. Since h is a uniform homeomorphism, there exist a finite subset $K' \subset Y$ and $\delta > 0$ such that for all $g', g'' \in C_p(Y)$ we have the implication

$$(|\,g'(y) - g''(y)| < \delta \ \text{ for all } y \in K') \Rightarrow |\,h(g')(x) - h(g'')(x)| < \varepsilon.$$

Let $g', g'' \in C_p(Y)$ and $|\, g'(y) - g''(y)| < \delta$ for all $y \in K$. Since Y is a Tychonoff space, there is $g \in C_p(Y)$ such that

$$g(y) = \begin{cases} g'(y) & \text{if } y \in K; \\ g''(y) & \text{if } y \in K' \setminus K. \end{cases}$$

Then $|\, g(y) - g''(y)| < \delta$ for all $y \in K'$, hence $|\, h(g)(x) - h(g'')(x)| < \varepsilon$. Now by the triangle inequality we obtain

$$|\, h(g')(x) - h(g'')(x)| \le |\, h(g')(x) - h(g)(x)| + |\, h(g)(x) - h(g'')(x)| < a(x, K, 0) + \varepsilon.$$

Passing to the supremum over all $g', g'' \in C_p(Y)$ such that $|\, g'(y) - g''(y)| < \delta$ for all $y \in K$, we have inequality $a(x, K, \delta) \le a(x, K, 0) + \varepsilon$, which implies that the function $\delta \mapsto a(x, K, \delta)$ is continuous at the point 0. Therefore there exists $\delta_0 > 0$ such that $a(x, K, \delta_0) < \infty$, hence $a(x, K, \delta) < \infty$ for all $\delta > 0$. $\square$

For any $x \in X$ we put $a(x) = a(x, K(x), 0)$. Using this notation we have the following simple assertions.

(K1) If $g', g'' \in C_p(Y)$ and $g'|_{K(x)} = g''|_{K(x)}$, then $|\, h(g')(x) - h(g'')(x)| \le a(x)$.

(K2) For any proper subset $K' \subset K(x)$ and any real b there exist functions $g', g'' \in C_p(Y)$ such that $g'|_{K'} = g''|_{K'}$ and $|\, h(g')(x) - h(g'')(x)| > b$.

Besides, this mapping surjectively maps the space X onto Y (see Lemma 3.2 on page 158), i.e., for any $y \in Y$ there exists $x \in X$ such that $y \in K(x)$.

For every $x \in X$ and every $\varepsilon > 0$ we define nonempty finite set $K_\varepsilon(x) \subset Y$ satisfying the following conditions:

(KE1) $a(x, K_\varepsilon(x), 0) \le \varepsilon$;

(KE2) $a(x, K', 0) > \varepsilon$ for every proper subset K' of $K_\varepsilon(x)$.

It is easy to check that such a set always exists. Indeed, since h is uniformly continuous, it follows that there exist $\delta > 0$ and a finite set $K \subset Y$ such that for all $g', g'' \in C_p(Y)$ we have the implication $(|\, g'(y) - g''(y)| < \delta$ for all $y \in K) \Rightarrow |\, h(g')(x) - h(g'')(x)| \le \varepsilon$. Then $a(x, K, 0) \le \varepsilon$. Reducing the set K until it satisfies the condition (KE2), we obtain the set $K_\varepsilon(x)$.

There can be several sets satisfying properties (KE1) and (KE2); then we denote by $K_\varepsilon(x)$ anyone of them. By the property 3 of $K(x)$ we have $K(x) \subset K_\varepsilon(x)$ for every $\varepsilon > 0$, and by the property 4 we have $K(x) = K_a(x)$ for any $a \ge a(x)$. Thus $K(x)$ is the smallest of all sets $K_\varepsilon(x)$.

The following lemma is analogous to result obtained by O. G. Okunev [4] for t-equivalence.

Lemma 1.3. *Let $x_0 \in X$, $\varepsilon > 0$, U is an open subset of Y such that $K(x_0) \cap U \neq \varnothing$. Then there is an open neighborhood V of x_0 such that $K_\varepsilon(x) \cap U \neq \varnothing$ for any $x \in V$.*

Proof. We can assume that $K(x_0) \cap U = \{y_0\}$. Put $K' = K(x_0) \setminus \{y_0\}$. By the property 2 of $K(x)$ there exist functions $g_1, g_2 \in C_p(Y)$ coinciding on K' such that $|\,h(g_1)(x_0) - h(g_2)(x_0)| > \varepsilon + a(x)$. Since Y is completely regular, it follows that there exists a function $g_0 \in C_p(Y)$ coinciding with g_1 on $Y \setminus U$ such that $g_0(y_0) = g_2(y_0)$. Then $g_0|_{K(x_0)} = g_2|_{K(x_0)}$ and $|\,h(g_0)(x_0) - h(g_2)(x_0)| \leq a(x)$. By the triangle inequality we obtain that

$$|\,h(g_1)(x_0) - h(g_0)(x_0)| \geq |\,h(g_1)(x_0) - h(g_2)(x_0)| - |\,h(g_0)(x_0) - h(g_2)(x_0)| > \varepsilon.$$

Let us prove that the set V defined by the formula $V = \{x \in X \colon |\,h(g_1)(x) - h(g_0)(x)| > \varepsilon\}$ is the required open neighborhood of x_0. Assume the contrary. Let $x \in V$ be a point such that $K_\varepsilon(x) \cap U = \varnothing$. Then g_1 coincides with g_0 on $K_\varepsilon(x)$. Therefore $|\,h(g_1)(x) - h(g_0)(x)| \leq \varepsilon$, a contradiction to the assumption that $x \in V$. $\square$

The last theorem yields the following corollaries.

Corollary 1.4. *Let $x_0 \in X$, $\varepsilon > 0$, $k \in \mathbb{N}$, and let U be an open subset of Y such that $|K(x_0) \cap U| \geq k$. Then there is an open neighborhood V of x_0 such that $|K_\varepsilon(x) \cap U| \geq k$ for all $x \in V$.*

The proof is trivial.

Corollary 1.5. *Let U be an open subset of Y. Then $K^{-1}(U)$ is a G_δ-set in X.*

Proof. Let $K^{-1}(U) \neq \varnothing$. Since $K(x) \subset K_m(x)$ for all $m \in \mathbb{N}$ and there is a natural number n such that $K(x) = K_n(x)$, it follows that $K^{-1}(U) = \bigcap_{m \in \mathbb{N}} K_m^{-1}(U)$. By Corollary 1.4 we have $K^{-1}(U) \subset \operatorname{Int} K_m^{-1}(U)$ for every $m \in \mathbb{N}$, consequently, $K^{-1}(U) = \bigcap_{m \in \mathbb{N}} \operatorname{Int} K_m^{-1}(U)$. $\square$

It is well known (see Lemma 3.5 on page 161) that every uniform homeomorphism h between C_p-spaces can be extended to a uniform homeomorphism between the spaces of all real-valued functions. We shall denote this new homeomorphism also by h.

Definition 1.6. *Fix a point $x \in X$, $\delta > 0$, and a finite subset $K \subset Y$, and put*

$$\bar{a}(x, K, \delta) = \sup\{|\, h(g')(x) - h(g'')(x)| : \\ g', g'' \in \mathbb{R}^Y,\ |g'(y) - g''(y)| < \delta \textit{ for all } y \in K\},$$

$$\bar{a}(x, K, 0) = \sup\{|\, h(g')(x) - h(g'')(x)| : \\ g', g'' \in \mathbb{R}^Y,\ g'(y) = g''(y) \textit{ for all } y \in K\}.$$

Lemma 1.7. *Let $h\colon \mathbb{R}^Y \to \mathbb{R}^X$ be a uniform homeomorphism such that $h(C_p(Y)) = C_p(X)$. Then $a(x, K, \delta) = \bar{a}(x, K, \delta)$ for all $x \in X$, any finite set $K \subset Y$, and $\delta \geq 0$.*

P r o o f. It follows from the definition that $a(x, K, \delta) \leq \bar{a}(x, K, \delta)$. Let us prove the reverse inequality. Let $\delta > 0$. Take $\varepsilon > 0$ and two functions $g_1, g_2 \in \mathbb{R}^Y$ such that

$$|\, g_1(y) - g_2(y)| < \delta \text{ for all } y \in K. \tag{1.1}$$

Since h is a uniform homeomorphism, it follows that there exist a finite set $K' \subset Y$ and $\Delta > 0$ such that for all $g', g'' \in \mathbb{R}^Y$ wehave the implication

$$(|\, g'(y) - g''(y)| < \Delta \ \text{ for all } y \in K') \Rightarrow |\, h(g')(x) - h(g'')(x)| < \varepsilon/2. \tag{1.2}$$

There are functions $g'_0, g''_0 \in C_p(Y)$ such that $g'_0|_{K \cup K'} \equiv g_1|_{K \cup K'}$ and $g''_0|_{K \cup K'} \equiv g_2|_{K \cup K'}$. Then $|\, g'_0(y) - g''_0(y)| < \delta$ for all $y \in K$. Observe that from (1.2) it follows that $|\, h(g_1)(x) - h(g'_0)(x)| < \varepsilon/2$ and $|\, h(g_2)(x) - h(g''_0)(x)| < \varepsilon/2$, and – by virtue of the triangle inequality – we have $a(x, K, \delta) \geq |\, h(g'_0)(x) - h(g''_0)(x)| > |\, h(g_1)(x) - h(g_2)(x)| - \varepsilon$. Passing to the supremum over all $g_1, g_2 \in \mathbb{R}^Y$ satisfying condition (1.1) we obtain inequality $a(x, K, \delta) \geq \bar{a}(x, K, \delta) - \varepsilon$. Since ε being an arbitrary positive number, this implies that $a(x, K, \delta) = \bar{a}(x, K, \delta)$. Equality $a(x, K, 0) = \bar{a}(x, K, 0)$ is proved analogously. □

2. Main result.

Theorem 2.1. *Let X and Y be u-equivalent, τ a cardinal not less than the continuum, and $l(X) \leq \tau$. Then $l(Y) \leq \tau$.*

P r o o f. Since any uniform homeomorphism between C_p-spaces can be extended to a uniform homeomorphism between the spaces of all real-valued functions, one can assume without loss of generality that there is a uniform homeomorphism h of $\mathbb{R}^Y$ onto $\mathbb{R}^X$ satisfying the following conditions:

1. $h(C_p(Y)) = C_p(X)$;
2. h takes zero function $0_Y \in \mathbb{R}^Y$ to zero function $0_X \in \mathbb{R}^X$.

To prove the theorem we shall need some notation.

Let $p\colon X \to Y$ be a set-valued mapping of X to Y and let $U \subset Y$ be an arbitrary set. Put

$$p^*(U) = \{x \in X : p(x) \subset U\}\,.$$

By $\mathcal{T}$ we shall denote the family of all open subsets of Y. Let $\mathcal{U}$ be an open cover of Y, τ an infinite cardinal. A cover $\mathcal{U}$ will be called τ-trivial if it contains a subcover of cardinality at most τ. Otherwise it will be called τ-nontrivial. This notion was introduced by A. Bouziad in [1]. Put

$$[\mathcal{U}]_\tau = \Big\{\bigcup \mathcal{U}' : \mathcal{U}' \subset \mathcal{U},\ |\mathcal{U}'| \le \tau\Big\}\,.$$

We say that the set A is an F_τ-subset of X if A can be represented as the union of some family, of cardinality at most τ, of closed subsets of X. The complements of F_τ-subsets will be called G_τ-subsets. If $\tau = \aleph_0$, then we shall write F_σ and G_δ instead of $F_{\aleph_0}$ and $G_{\aleph_0}$ respectively. The symbol $\mathcal{F}_\tau$ denotes the family of all F_τ-subsets of X, $\mathcal{G}_\tau$ is a family of all G_τ-subsets of X. The family of all subsets A of X such that $l(A) \le \tau$ will be denoted by $\mathcal{L}_\tau$.

Let $l(X) \le \tau$, where $\tau \ge c$. Assume that $l(Y) > \tau$ to obtain a contradiction. It means that there exists τ-nontrivial open cover $\mathcal{U}$ of Y. Without loss of generality we can assume that $\mathcal{U}$ is closed under the operation of finite union and $\mathcal{U} \subset \mathcal{B}$, where $\mathcal{B}$ is a base of Y consisting of all functionally open subsets of Y. It is well known that the family $\mathcal{B}$ is also closed under the operation of finite union (see [2], page 43).

Define a mapping

$$U\colon \operatorname{Fin} \mathcal{F}_\tau \to [\mathcal{U}]_\tau\,, \qquad U = U(\mathcal{F}), \qquad \text{where} \qquad \mathcal{F} \in \operatorname{Fin} \mathcal{F}_\tau,$$

using set-valued mappings defined in the previous section. For any $x \in X$ put $\rho(x) = |\,K(x)|$. For every set $F \subset X$ we define a number

$$\rho(F) = \min\{\rho(x) : x \in F\}\,,$$

which will be called the level of the set F.

Further, for any $U \in \mathcal{T}$ and any natural numbers k and m put

$$U_m^{[k]} = \operatorname{Int}\{\, x \in X \colon |\, K_m(x) \cap U \,| \geq k \,\}\,.$$

Let $\mathcal{F} = \{F_1, \dots, F_n\} \subset \mathcal{F}_\tau$. For any nonempty set $A \subset \{1, \dots, n\}$ we put

$$F_A = \bigcap_{i \in A} F_i, \qquad \overline{\mathcal{F}} = \{\, F_A \colon A \subset \{1, \dots, n\},\ F_A \neq \varnothing \,\}.$$

Let $F \in \overline{\mathcal{F}}$, $m \in \mathbb{N}$ and $k = \rho(F)$. Then the family

$$\mathcal{U}_m^{[k]} = \left\{ U_m^{[k]} \colon U \in \mathcal{U} \right\}$$

is an open cover of F. Indeed, since the family $\mathcal{U}$ is closed under the operation of finite union, it follows that for every $x_0 \in F$ there is $U \in \mathcal{U}$ such that $K(x_0) \subset U$. As $\rho(F) = k$, it follows that $|K(x_0) \cap U| \geq k$ and by Corollary 1.4 there exists an open neighborhood V of x_0 such that $|K_m(x) \cap U| \geq k$ for all $x \in V$. Then $x_0 \in V \subset U_m^{[k]}$, hence $\mathcal{U}_m^{[k]}$ is an open cover of F. From the condition $l(X) \leq \tau$ it follows that $F \in \mathcal{L}_\tau$; therefore the cover $\mathcal{U}_m^{[k]}$ contains a subcover $\left\{ U_m^{[k]} \colon U \in \mathcal{U}_{F,m} \right\}$ of F, where $\mathcal{U}_{F,m} \subset \mathcal{U}$ and $|\mathcal{U}_{F,m}| \leq \tau$. Put

$$U(\mathcal{F}) = \bigcup_{F \in \overline{\mathcal{F}}} \bigcup_{m \in \mathbb{N}} \left(\bigcup \mathcal{U}_{F,m} \right).$$

Obviously, $U(\mathcal{F}) \in [\mathcal{U}]_\tau$, and if $\mathcal{F}_1 \subset \mathcal{F}_2$, then $U(\mathcal{F}_1) \subset U(\mathcal{F}_2)$. The mapping U we shall call the constructor. A similar construction was used by N. V. Velichko in [5].

We note one important property of the constructor.

(*) For every $\mathcal{F} \in \operatorname{Fin} \mathcal{F}_\tau$, any $F \in \overline{\mathcal{F}}$, and any $x \in F$ the following inequality holds:

$$|\, K(x) \cap U\,(\mathcal{F})| \geq \rho(F). \tag{2.1}$$

Indeed, for any $x \in F$ there exist a natural number m and a set $U \in \mathcal{U}_{F,m}$ such that $K_m(x) = K(x)$ and $x \in U_m^{[k]}$; hence $|\, K(x) \cap U\,(\mathcal{F})| \geq |\, K(x) \cap U| \geq k = \rho(F)$.

Let us recall some important properties of the set-valued mappings K and K_m defined in the previous section.

(P1) If $g', g'' \in \mathbb{R}^Y$ and $g'|_{K_m(x)} = g''|_{K_m(x)}$, then $|h(g')(x) - h(g'')(x)| \leq m$. In particular, if $g'|_{K_m(x)} \equiv 0$, then $|h(g')(x)| \leq m$.

(P2) If $g', g'' \in \mathbb{R}^Y$ and $g'|_{K(x)} = g''|_{K(x)}$, then $|h(g')(x) - h(g'')(x)| \leq a(x) < \infty$. In particular, if $g'|_{K(x)} \equiv 0$, then $|h(g')(x)| \leq a(x) < \infty$.

For each $V \subset Y$ consider the function $e_V \in \mathbb{R}^Y$ defined by the formula

$$e_V(y) = \begin{cases} 0, & y \in V, \\ 1, & y \notin V. \end{cases}$$

Denote by $\mathcal{C}$ the family of all functionally closed subsets of Y. Every functionally open set $V \subset Y$ admits a decomposition

$$V = \bigcup_{n \in \mathbb{N}} F_n\,, \quad \text{where } F_n \in \mathcal{C} \text{ and } F_n \subset F_{n+1} \quad \text{for all } n \in \mathbb{N} \tag{2.2}$$

(see Lemma 3.4 on page 160). Further, by decomposition of functionally open set V we mean a sequence $(F_n)_{n\in\mathbb{N}}$ satisfying condition (2.2). If there is a decomposition $(F_n)_{n\in\mathbb{N}}$ of V satisfying the following condition:

$$K_1^*(V) \setminus K_1^*(F_n) \neq \varnothing \quad \text{for all } n \in \mathbb{N}\,, \tag{2.3}$$

then we say that the set V is *adequate*. A similar notion was introduced by A. Bouziad in [1].

For every open set $V \in \mathcal{T}$ put

$$G(V) = \left\{ x \in X \colon \sup_{m\in\mathbb{N}} |h(me_V)(x)| < \infty \right\},$$

$$F(V) = \left\{ x \in X \colon \sup_{m\in\mathbb{N}} |h(me_V)(x)| = \infty \right\}.$$

Analogous mappings were used by A. Bouziad in [1].

Lemma 2.2. *The mapping G has the following properties:*

(S1) $K^*(V) \subset G(V)$ *for any* $V \in \mathcal{T}$;

(S2) *For any expanding sequence* $(U_n)_{n\in\mathbb{N}}$ *of the sets* $U_n \in \mathcal{T}$ *such that*

$$X = \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} G\,(U_n) \tag{2.4}$$

the following condition holds:

$$Y = \bigcup_{n\in\mathbb{N}} U_n\,.$$

P r o o f. Let us verify that condition (S1) is satisfied. Take $V \in \mathcal{T}$ and $x \in K^*(V)$. Then $K(x) \subset V$, hence $me_V|_{K(x)} \equiv 0$ for any natural number m and by (P2) we have $|\,h(me_V)(x)| \leq a(x) < \infty$, therefore $\sup_{m\in\mathbb{N}} |\,h(me_V)(x)| \leq a(x) < \infty$, which implies that $x \in G(V)$.

Let us show that condition (S2) is fulfilled. Let $(U_n)_{n\in\mathbb{N}}$ be an expanding sequence of the sets $U_n \in \mathcal{T}$ such that equality (2.4) is valid. Assume that $Y \neq \bigcup_{n\in\mathbb{N}} U_n$. Put $U = \bigcup_{n\in\mathbb{N}} U_n$. Take $y \in Y \setminus U$. Choose a finite subset $K' = \{x_1, \ldots, x_p\} \subset X$ and $\delta > 0$ such that for any two functions $f', f'' \in \mathbb{R}^X$ the following implication holds:

$$\big(\,|f'(x_i) - f''(x_i)| \leq \delta \quad \text{for all } i \in \overline{1,p}\,\big) \Rightarrow |\,h^{-1}(f')(y) - h^{-1}(f'')(y)| < 1.$$

Such a choice is possible because the mapping h^{-1} is uniformly continuous. Then, as shown in [3], for any two functions $f', f'' \in \mathbb{R}^X$ and every natural number n the following implication holds:

$$\big(\,|f'(x_i) - f''(x_i)| \leq n\delta \ \text{ for all } i \in \overline{1,p}\,\big) \Rightarrow |\,h^{-1}(f')(y) - h^{-1}(f'')(y)| < n.$$

In particular,

$$\big(\,|\,h(g)(x_i)| \leq n\delta \ \text{ for all } i \in \overline{1,p}\,\big) \Rightarrow |\,g(y)| < n \tag{2.5}$$

for any $g \in \mathbb{R}^Y$.

From equality (2.4) it follows that there is a natural number N such that $x_i \in G\,(U_N)$ for all $i \in \overline{1,p}$. Put

$$M = \max_{i\in\overline{1,p}} \sup_{m\in\mathbb{N}} |h(me_{U_N})(x_i)|.$$

Obviously, $M < \infty$. Pick a natural number $n \in \mathbb{N}$ such that $n \geq M/\delta$. Then $|\,h(ne_{U_N})(x_i)| \leq M \leq n\delta$ for all $i \in \overline{1,p}$. From this inequality and condition (2.5) it follows that $|ne_{U_N}(y)| < n$, hence $|e_{U_N}(y)| < 1$, therefore $y \in U_N \subset U$. Thus we obtain a contradiction. $\square$

Lemma 2.3. *Let $\{U_t\}_{t\in T} \subset \mathcal{U}$ and $|T| \le \tau$. Then there is a family $\{V_s\}_{s\in S} \subset [\mathcal{U}]_\tau$ closed under the operation of finite union and satisfying the following conditions:*

1. $|S| \le \tau$*;*
2. *each set V_s is adequate;*
3. $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$.

Proof. Let $V_0 = \bigcup_{t\in T} U_t$. Since the cover $\mathcal{U}$ is τ-nontrivial, there exists $y_1 \in Y \setminus V_0$. Choose $x_1 \in X$ such that $y_1 \in K_1(x_1)$, i.e., $K_1(x_1) \not\subseteq V_0$ (such an element exists since the mapping $x \mapsto K(x)$ is surjective), and choose a set $V_1 \in \mathcal{U}$ such that $K_1(x_1) \subset V_1$ (such a set exists since the set $K_1(x_1)$ is finite and the family $\mathcal{U}$ is closed under the operation of finite union). Assume that $x_1, \dots, x_k$ and $V_1, \dots, V_k$ are already chosen, where $k \in \mathbb{N}$. The set $Y \setminus \bigcup_{i=0}^{k} V_i$ is nonempty, hence there is an element $x_{k+1} \in X$ such that $K_1(x_{k+1}) \not\subseteq \bigcup_{i=0}^{k} V_i$ and there is a set $V_{k+1} \in \mathcal{U}$ such that $K_1(x_{k+1}) \subset V_{k+1}$. We obtain two sequences $(x_n)_{n\in\mathbb{N}} \subset X$ and $(V_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ such that $K_1(x_n) \not\subseteq \bigcup_{i=0}^{n-1} V_i$, $V_n \in \mathcal{U}$, and $K_1(x_n) \subset V_n$ for any natural number n. Put $V = \bigcup_{n\in\mathbb{N}} V_n$. Let $(W_s)_{s\in S}$ be the family of all finite unions of sets in $(U_t)_{t\in T}$. For each $s \in S$ put $V_s = W_s \cup V$. Clearly, the family $(V_s)_{s\in S} \subset [\mathcal{U}]_\tau$ is closed under the operation of finite union, $|S| \le \tau$, and $\bigcup_{t\in T} U_t \subset \bigcup_{s\in S} V_s$. It remains to verify that each set V_s is adequate. Let $s \in S$. Fix a decomposition $(F_n^s)_{n\in\mathbb{N}}$ of the set W_s and decomposition $\left(F_n^k\right)_{n\in\mathbb{N}}$ of the set V_k, where $k \in \mathbb{N}$. The sequence $(G_n^s)_{n\in\mathbb{N}}$, where $G_n^s = F_n^s \cup F_n^1 \cup \ldots \cup F_n^n$, is a required decomposition of the set V_s, since $(x_n)_{n\in\mathbb{N}} \subset K_1^*(V_s)$ and $x_{n+1} \notin K_1^*(G_n^s)$ for all $n \in \mathbb{N}$. $\square$

Lemma 2.4. *Let $\{V_s\}_{s\in S}$ be a family of adequate functionally open subsets of Y closed under the operation of finite union $|S| \le \tau$. Then $F\left(\bigcup_{s\in S} V_s\right)$ is an F_τ-subset of X.*

Proof. Put $V = \bigcup_{s\in S} V_s$. Let $(F_n^s)_{n\in\mathbb{N}}$ be a decomposition of V_s satisfying conditions $F_n^s \in \mathcal{C}$, $F_n^s \subset F_{n+1}^s$, and $K_1^*(V_s) \setminus K_1^*(F_n^s) \neq \varnothing$ for all $n \in \mathbb{N}$. For any natural number n and any $s \in S$ we can find a function $g_n^s \in C_p(Y)$ (see Lemma 3.5 on page 161) such that

$$g_n^s|_{F_n^s} \equiv 0\,, \qquad g_n^s|_{Y\setminus V_s} \equiv 1.$$

For any $x \in K_1^*(V_s)$ and $k, n \in \mathbb{N}$ put

$$U_{k,\,n}^s(x) = \left\{x' \in X\colon \left|h\left(ng^s_{k+N(x,\,s)}\right)(x') - h\left(ng^s_{k+N(x,\,s)}\right)(x)\right| < k\right\},$$

where $N(x,s)$ is the smallest natural number N such that $K_1(x) \subset F_N^s$. Then $U_{k,\,n}^s(x)$ is an open neighborhood of the point x in X. Put

$$A_s = \bigcap_{m\in\mathbb{N}} \bigcup_{k\geq m} \bigcap_{n\in\mathbb{N}} \bigcup_{x\in K_1^*(V_s)} U_{k,\,n}^s(x)\,, \quad B_s = \{x \in X\colon K(x) \cap (V \setminus V_s) \neq \varnothing\}\,,$$

$$A = \bigcap_{s\in S} (A_s \cup B_s).$$

Since each set $\bigcup_{x\in K_1^*(V_s)} U_{k,\,n}^s(x)$ is open in X, by Corollary 3.7 on page 161 we have that $\bigcup_{k\geq m}\bigcap_{n\in\mathbb{N}}\bigcup_{x\in K_1^*(V_s)} U_{k,\,n}^s(x)$ is a G_c-subset of X for any natural number n, which implies that A_s is a G_c-set. Since B_s is a G_δ-subset of X (see Lemma 3.8 on page 161), it follows that A is a G_τ-subset of X. Here we have used the fact that $\tau \geq c$. We shall prove that $G(V) = A$. Since $F(V) = X \setminus G(V)$, this will be sufficient to prove the lemma. We first prove that

$$F(V) \subset X \setminus A. \tag{2.6}$$

Take $x' \in F(V)$. Since $K(x')$ is a finite set and the family $\{V_s\}_{s\in S}$ is closed under the operation of finite union, there exists $s \in S$ such that $K(x') \cap V \subset V_s$, i.e., $x' \notin B_s$. It remains to prove that $x' \notin A_s$. There exists a natural number m_0 satisfying the condition $K(x') \cap V \subset F_{m_0}^s$. Then

$$e_V|_{K(x')} = e_{V_s}|_{K(x')} = g_n^s|_{K(x')} \tag{2.7}$$

for any $n \geq m_0$. Since $x' \in F(V)$, for any $k \in \mathbb{N}$ there is a natural number n_k such that

$$\left|h\left(n_k e_V\right)(x')\right| \geq k + a(x') + 1. \tag{2.8}$$

Take an arbitrary natural number $k \geq m_0$. We verify that $x' \notin \bigcup_{x\in K_1^*(V_s)} U_{k,n_k}^s(x)$.

From (2.7) and (P2) it follows that $|\,h\left(n_k e_V\right)(x') - h\left(n_k g_n^s\right)(x')| \leq a(x')$ for any natural numbers $n, k \geq m_0$ and this together with (2.8) gives the inequality $|\,h\left(n_k g_n^s\right)(x')| \geq k+1$. Take an arbitrary $x \in K_1^*(V_s)$. It remains to show that $x' \notin U^s_{k,\,n_k}(x)$. Since $g^s_{k+N(x,\,s)}|_{K_1(x)} \equiv 0$, from (P1) it follows that $|\, h(n_k g^s_{k+N(x,\,s)})(x)| \leq 1$. Then

$$\left| h \left(n_k g^s_{k+N(x,\,s)} \right) (x') - h \left(n_k g^s_{k+N(x,\,s)} \right) (x) \right|$$

$$\geq \left| h \left(n_k g^s_{k+N(x,\,s)} \right) (x') \right| - \left| h \left(n_k g^s_{k+N(x,\,s)} \right) (x) \right| \geq (k+1) - 1 = k.$$

Hence, $x' \notin U^s_{k,\,n_k}(x)$. Inclusion (2.6) is proved.

Let us prove the reverse inclusion $X \setminus A \subset F(V)$. Let $x' \notin A$. We shall show that $x' \in F(V)$. Choose $s \in S$ such that $x' \notin A_s \cup B_s$. Then $K(x') \cap V \subset V_s$. Fix $m_0 \in \mathbb{N}$ such that $x' \notin \bigcup_{k \geq m_0} \bigcap_{n \in \mathbb{N}} \bigcup_{x \in K_1^*(V_s)} U^s_{k,\,n}(x)$ and take an arbitrary natural number $k \geq m_0$. Then there is $n_k \in \mathbb{N}$ such that $x' \notin \bigcup_{x \in K_1^*(V_s)} U^s_{k,\,n_k}(x)$. Choose $q \in \mathbb{N}$ such that $K(x') \cap V \subset F^s_q$ and an element $x_0 \in K_1^*(V_s)$ satisfying the condition $K_1(x_0) \nsubseteq F^s_q$. Such an element exists because the set V_s is adequate. Then $N(x_0, s) > q$ and

$$K(x') \cap V = K_1(x') \cap V_s \subset F^s_q \subset F^s_{k+N(x_0,\,s)}.$$

Put $i = k + N(x_0, s)$. Since $x' \notin U^s_{k,\,n_k}(x_0)$, we have $|h\,(n_k g^s_i)\,(x') - h\,(n_k g^s_i)\,(x_0)| \geq k$. Besides, $|h\,(n_k g^s_i)\,(x_0)| \leq 1$. Hence, by the triangle inequality we obtain that

$$\left| h\,(n_k g^s_i)\,(x') \right| \geq k - 1.$$

Since $e_V|_{K(x')} = e_{V_s}|_{K(x')} = g^s_i|_{K(x')}$, we have $|\, h(n_k g^s_i)(x') - h(n_k e_V)(x')| \leq a(x')$. Then, again applying the triangle inequality we obtain

$$|\, h(n_k e_V)(x')| \geq |\, h(n_k g^s_i)(x')| - |\, h(n_k g^s_i)(x') - h(n_k e_V)(x')| \geq k - 1 - a(x'),$$

hence, $\sup_{m \in \mathbb{N}} |\, h(m e_V)(x')| = \infty$. $\square$

Lemmas 2.4 and 2.3 yield the following corollary.

Corollary 2.5. *For any $U \in [\mathcal{U}]_\tau$ there exists $V \in [\mathcal{U}]_\tau$ such that $U \subset V$ and $F(V)$ is an F_τ-subset of X.*

We shall now construct an expanding sequence $(V_n)_{n \in \mathbb{N}}$ such that $V_n \in [\mathcal{U}]_\tau$. Simultaneously with it we shall construct a sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \in \operatorname{Fin} \mathcal{F}_\tau$ and $\mathcal{F}_{n'} \subset \mathcal{F}_{n''}$ for any two natural numbers $n' < n''$.

Let $\mathcal{F}_0 = \{X\}$. Choose a set $V_1 \in [\mathcal{U}]_\tau$ such that

$$U(\mathcal{F}_0) \subset V_1 \qquad \text{and} \qquad F(V_1) \in \mathcal{F}_\tau$$

(it is possible by Corollary 2.5), and put $\mathcal{F}_1 = \{X,\ F(V_1)\}$. Choose a set $V_2 \in [\mathcal{U}]_\tau$ such that

$$V_1 \cup U(\mathcal{F}_1) \subset V_2 \qquad \text{and} \qquad F(V_2) \in \mathcal{F}_\tau .$$

Assume that we have already defined the sets $V_i \in [\mathcal{U}]_\tau$ and $\mathcal{F}_i \in \operatorname{Fin} \mathcal{F}_\tau$ for every natural number $i \leq k$ satisfying the following conditions:

1. $F(V_i) \in \mathcal{F}_\tau, \qquad 1 \leq i \leq k;$
2. $V_i \cup U(\mathcal{F}_i) \subset V_{i+1}, \qquad 1 \leq i \leq k-1$, where $\mathcal{F}_i = \{X, F(V_1), \ldots, F(V_i)\}$, $1 \leq i \leq k$.

Choose a set $V_{k+1} \in [\mathcal{U}]_\tau$ satisfying the following conditions:

$$V_k \cup U(\mathcal{F}_k) \subset V_{k+1} \qquad \text{and} \qquad F(V_{k+1}) \in \mathcal{F}_\tau . \tag{2.9}$$

Put $\mathcal{F}_{k+1} = \{X, F(V_1), \ldots, F(V_{k+1})\}$. The sequences $(V_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ are defined.

We shall prove by induction with respect to n the following assertion.

Assertion 2.6. *For any natural number n and each set $\{j_1, \ldots, j_k\} \subset \{1, \ldots, n\}$ such that $F(V_{j_1}) \cap \ldots \cap F(V_{j_k}) \neq \varnothing$ the following inequality holds:*

$$\rho(F(V_{j_1}) \cap \ldots \cap F(V_{j_k})) \geq k+1. \tag{2.10}$$

Proof. We shall show that $\rho(F(V_n)) \geq 2$. For any $x \in X$ by inequality (2.1) we have

$$|K(x) \cap V_n| \geq |K(x) \cap V_1| \geq |K(x) \cap U(\mathcal{F}_0)| \geq \rho(X) \geq 1.$$

Therefore, if $\rho(x) = 1$ for some $x \in X$, then $K(x) \subset V_n$, hence, by (S1) we have $x \notin F(V_n)$. This implies that $\rho(F(V_n)) \geq 2$. In particular, this yields that the assertion is valid for $n = 1$.

Assume that Assertion 2.6 holds for every natural number $n \leq N$. We shall prove that it holds for $n = N+1$. It suffice to show that for each subset $\{j_1, \ldots, j_k\} \subset \{1, \ldots, N\}$ such that $F = F(V_{j_1}) \cap \ldots \cap F(V_{j_k}) \cap F(V_{N+1}) \neq \varnothing$

the following inequality holds: $\rho(F) \geq k+2$. Put $F' = F(V_{j_1}) \cap \ldots \cap F(V_{j_k})$, then $F = F' \cap F(V_{N+1})$. By induction hypothesis we have inequality $\rho(F') \geq k+1$. Assume that $\rho(F) = k+1$ to obtain a contradiction.

Take an element $x \in F$ such that $|K(x)| = k+1$. Since $F' \in \overline{\mathcal{F}_N}$, we see that from (2.9) and (2.1) it follows that

$$|K(x) \cap V_{N+1}| \geq |K(x) \cap U(\mathcal{F}_N)| \geq \rho(F') \geq k+1.$$

Hence, $K(x) \subset V_{N+1}$ and condition (S2) implies that $x \notin F(V_{N+1})$. Therefore $x \notin F$. This contradiction completes the proof of Assertion 2.6.

In particular, inequality (2.10) implies that for any $x \in X$ there exists a natural number k such that $x \notin F(V_n)$ for all $n > k$, i.e., that $x \in G(V_n)$. In other words, equality (2.4) holds. By Lemma 2.2 we obtain $Y = \bigcup_{n \in \mathbb{N}} V_n$. Since $V_n \in [\mathcal{U}]_\tau$ for any $n \in \mathbb{N}$, we see that the cover $\mathcal{U}$ of Y is τ-trivial, a contradiction. Hence, $l(Y) \leq \tau$. □

Corollary 2.7. *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic, and let $l(X), l(Y) \geq c$. Then $l(X) = l(Y)$.*

Corollary 2.8. *Let the spaces $C_p(X)$ and $C_p(Y)$ be uniformly homeomorphic. Then $l(X) \leq c$ if and only if $l(Y) \leq c$.*

The statement of Theorem 0.1 follows from Corollaries 2.7 and 2.8.

Problem 2.9 *Are there spaces X and Y such that $l(X) = c$, $l(Y) < c$ and $C_p(X)$ is uniformly homeomorphic to $C_p(Y)$?*

3. Auxiliary statements used in the proof.

Theorem 3.1. *Let $h\colon C_p(Y) \to C_p(X)$ be a uniform homeomorphism. Then there is a uniform homeomorphism $\bar{h}\colon \mathbb{R}^Y \to \mathbb{R}^X$ such that $\bar{h}(g) = h(g)$ for all $g \in C_p(Y)$.*

Proof. Let $\widetilde{K}_n(x) = \bigcup_{m=1}^{n} K_{1/m}(x)$, $\widetilde{K}(x) = \bigcup_{m=1}^{\infty} K_{1/m}(x)$, where $x \in X$. For the mapping $H = h^{-1}\colon C_p(X) \to C_p(Y)$ we define such mappings as defined in section 1 for h. For any $y \in Y$, $\delta > 0$, and any finite subset $L \subset X$ put

$$b(y, L, \delta) = \sup\{|H(f')(y) - H(f'')(y)| : \\ f', f'' \in C_p(X),\ |f'(x) - f''(x)| < \delta \text{ for all } x \in L\}.$$

We also put

$$b(y, L, 0) = \sup\{|\, H(f')(y) - H(f'')(y)| : \\ f', f'' \in C_p(X),\ f'(x) = f''(x) \text{ for all } x \in L\}.$$

As in the case of the mapping h, for every $y \in Y$ there exist finite sets $L(y) \subset X$ and $L_\varepsilon(y) \subset X$ for any $\varepsilon > 0$ satisfying the following conditions:

1. $b(y, L(y), \delta) < \infty$ for all $\delta \geq 0$;
2. $b(y, L', \delta) = \infty$ for all $\delta \geq 0$, where L' is a proper subset of $L(y)$;
3. If $b(y, L, \delta) < \infty$ for some finite set $L \subset X$ and $\delta \geq 0$, then $L(y) \subset L$;
4. $b(y, L_\varepsilon(y), 0) \leq \varepsilon$;
5. $b(y, L', 0) > \varepsilon$, where L' is a proper subset of $L_\varepsilon(y)$;
6. $L(y) \subset L_\varepsilon(y)$.

Let $\widetilde{L}_n(y) = \bigcup_{m=1}^{n} L_{1/m}(y)$, $\widetilde{L}(y) = \bigcup_{m=1}^{\infty} L_{1/m}(y)$, where $y \in Y$.

For the proof we need two lemmas.

Lemma 3.2. *$y \in \bigcup_{x \in L(y)} K(x)$ for any $y \in Y$.*

Proof. Let $K = \bigcup_{x \in L(y)} K(x)$. Assume that $y \notin K$ to obtain a contradiction. Let $\delta = \max\{a(x) \colon x \in L(y)\}$, $b = b(y, L(y), \delta)$. Take a function $g \in C_p(Y)$ such that $g|_K \equiv 0$ and $g(y) = b + 1$. Since $g|_{K(x)} \equiv 0$, we have $|\, h(g)(x)| \leq a(x) \leq \delta$ for any $x \in L(y)$. Then $b + 1 = |g(y)| \leq b(y, L(y), \delta) = b$. This contradiction completes the proof. □

We now define a mapping $\bar{h} \colon \mathbb{R}^Y \to \mathbb{R}^X$. Let $g \in \mathbb{R}^Y$ and $x \in X$. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of continuous functions on Y such that $g_n|_{\widetilde{K}_n(x)} = g|_{\widetilde{K}_n(x)}$ for each $n \geq n_0$, where n_0 is some natural number. We shall prove that the sequence $(h(g_n)(x))_{n \in \mathbb{N}}$ has a limit. Take $\varepsilon > 0$ and put $N = \max([1/\varepsilon] + 1, n_0)$, where $[x]$ denotes the integer part of x. Then $g_n|_{\widetilde{K}_N(x)} = g_m|_{\widetilde{K}_N(x)}$ for all $n, m \geq N$, hence, $|\, h(g_n)(x) - h(g_m)(x)| \leq 1/N < \varepsilon$. We obtain that the sequence $(h(g_n)(x))_{n \in \mathbb{N}}$ is fundamental (Cauchy sequence), hence it has a limit. We define a mapping $\bar{h}$ by the formula

$$\bar{h}(g)(x) = \lim_{n \to \infty} h(g_n)(x).$$

We have to prove that the definition does not depend on the choice of the sequence $(g_n)_{n \in \mathbb{N}}$. Let $(g'_n)_{n \in \mathbb{N}}$ be another sequence of continuous functions on Y such that $g'_n|_{\widetilde{K}_n(x)} = g|_{\widetilde{K}_n(x)}$ starting from some n_1, and let $a = \lim_{n \to \infty} h(g_n)(x)$,

$b = \lim_{n\to\infty} h(g'_n)(x)$. From the sequences $\{g_n\}$ and $\{g'_n\}$, we construct another sequence $\{g''_n\}$ defined by the formula

$$g''_n = \begin{cases} g_n & \text{if } n \text{ is odd;} \\ g'_n & \text{if } n \text{ is even.} \end{cases}$$

As shown above, there is a limit of the sequence $(h(g''_n)(x))_{n\in\mathbb{N}}$ which we denote by c. Then

$$c = \lim_{n\to\infty} h(g''_n)(x) = \lim_{n\to\infty} h(g''_{2n})(x) = \lim_{n\to\infty} h(g''_{2n-1}),$$

which implies that $a = b = c$. Obviously, if $g \in C_p(Y)$, then $\bar{h}(g) = h(g)$.

We now define a mapping $\bar{H}\colon \mathbb{R}^X \to \mathbb{R}^Y$. Let $f \in \mathbb{R}^X$ and $y \in Y$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on X such that $f_n|_{\widetilde{L}_n(y)} = f|_{\widetilde{L}_n(y)}$ starting from some n_0. Similarly, we can prove that there is a limit of the sequence $(h^{-1}(f_n)(y))_{n\in\mathbb{N}}$. Consider the mapping H defined by the formula $\bar{H}(f)(y) = \lim_{n\to\infty} h^{-1}(f_n)(y)$. It can be proved analogously that the definition is correct and $\bar{H}(f) = h^{-1}(f)$ for all $f \in C_p(X)$.

Lemma 3.3. *The mappings $\bar{h}\colon \mathbb{R}^Y \to \mathbb{R}^X$ and $\bar{H}\colon \mathbb{R}^X \to \mathbb{R}^Y$ are uniformly continuous.*

Proof. Take $x \in X$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > 4/\varepsilon$. Then for each natural number $n \ge N$ we have $a(x, \widetilde{K}_n(x), 0) \le 1/N < \varepsilon/4$. Since the mapping $\delta \mapsto a(x, K, \delta)$ is continuous at zero, there exists $\delta > 0$ such that

$$a(x, \widetilde{K}_N(x), \delta) < \varepsilon/2\,. \tag{3.1}$$

Let $g', g'' \in \mathbb{R}^Y$ and $|g'(y) - g''(y)| < \delta$ for any $y \in \widetilde{K}_N(x)$. We shall consider the sequences $(g'_n)_{n\in\mathbb{N}}, (g''_n)_{n\in\mathbb{N}} \subset C_p(Y)$ such that $g'_n|_{\widetilde{K}_n(x)} = g'|_{\widetilde{K}_n(x)}$ and $g''_n|_{\widetilde{K}_n(x)} = g''|_{\widetilde{K}_n(x)}$ for all $n \in \mathbb{N}$. Then $|\,h(g'_N)(x) - h(g'_n)(x)| \le 1/N < \varepsilon/4$ and $|\,h(g''_N)(x) - h(g''_n)(x)| \le 1/N < \varepsilon/4$ for all $n \ge N$. It is clear that $\lim_{n\to\infty} h(g'_n)(x) = \bar{h}(g')(x)$ and $\lim_{n\to\infty} h(g''_n)(x) = \bar{h}(g'')(x)$. Hence, passing to the limit in the last inequalities as $n \to \infty$, we obtain inequalities $|\,h(g'_N)(x) - \bar{h}(g')(x)| < \varepsilon/4$ and $|\,h(g''_N)(x) - \bar{h}(g'')(x)| < \varepsilon/4$. In addition, $|\,g'_N(y) - g''_N(y)| < \delta$ for all $y \in \widetilde{K}_N(x)$, therefore, from (3.1) it follows that $|\,h(g'_N)(x) - h(g''_N)(x)| < \varepsilon/2$. Then

$$\begin{aligned} &|\,\bar{h}(g')(x) - \bar{h}(g'')(x)| \\ &= |\,(\bar{h}(g')(x) - h(g'_N)(x)) + (h(g'_N)(x) - h(g''_N)(x)) + (h(g''_N)(x) - \bar{h}(g'')(x))| \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \end{aligned}$$

The proof for $\bar{H}$ is analogous. □

We now prove that $\bar{H} = \bar{h}^{-1}$. Let $g \in \mathbb{R}^Y$, $y \in Y$. We shall show that $\bar{H}(\bar{h}(g))(y) = g(y)$. For any natural numbers n, m put

$$\widetilde{K}_{n,m}(y) = \bigcup_{x \in \widetilde{L}_n(y)} \widetilde{K}_m(x).$$

Take a sequence $(f_n)_{n\in\mathbb{N}} \subset C_p(X)$ such that $f_n|_{\widetilde{L}_n(y)} = \bar{h}(g)|_{\widetilde{L}_n(y)}$ for every natural number n. Put $g_n = h^{-1}(f_n) \in C_p(Y)$. Then $\bar{H}(\bar{h}(g))(y) = \lim_{n\to\infty} g_n(y)$. Since the mapping $\delta \mapsto b(y, L, \delta)$ is continuous at zero, for any natural number n there is $\delta_n > 0$ such that for any two functions $g', g'' \in C_p(Y)$ the following implication holds:

$$\text{(3.2)} \quad \Big(|\, h(g')(x) - h(g'')(x)| < \delta_n \text{ for all } x \in \widetilde{L}_n(y)\Big) \Rightarrow |\, g'(y) - g''(y)| < 2/n\,.$$

Take a sequence $(g'_m)_{m\in\mathbb{N}} \subset C_p(Y)$ such that $g'_m|_{\widetilde{K}_{m,m}(y)} = g|_{\widetilde{K}_{m,m}(y)}$ for all natural number m. Then for each $x \in \widetilde{L}(y)$ there is natural number m_x such that for any $m \geq m_x$ we have $g'_m|_{\widetilde{K}_m(x)} = g|_{\widetilde{K}_m(x)}$; hence, $\lim_{m\to\infty} h(g'_m)(x) = \bar{h}(g)(x)$ for each $x \in \widetilde{L}(y)$. Therefore, for any natural number n there is $m_n \in \mathbb{N}$ such that $|\, h(g'_{m_n})(x) - \bar{h}(g)(x)| < \delta_n$ for each $x \in \widetilde{L}_n(y)$; hence,

$$|\, h(g'_{m_n})(x) - h(g_n)(x)| = |\, h(g'_{m_n})(x) - f_n(x)| = |\, h(g'_{m_n})(x) - \bar{h}(g)(x)| < \delta_n$$

for each $x \in \widetilde{L}_n(y)$. From (3.2) it follows that $|\, g'_{m_n}(y) - g_n(y)| < 2/n$. Since $y \in \widetilde{K}_{1,1}(y)$ by Lemma 3.2, we obtain the equality $g'_m(y) = g(y)$ for every natural number m, which implies that $|\, g(y) - g_n(y)| < 2/n$. Passing to the limit in this inequality as $n \to \infty$, we obtain that $g(y) = \bar{H}(\bar{h}(g))(y)$. It can be proved analogously that $\bar{h}(\bar{H}(f))) = f$ for any $f \in \mathbb{R}^X$, which implies that $\bar{H} = \bar{h}^{-1}$. This completes the proof of Theorem 3.1. □

Lemma 3.4. *Let U be a functionally open subset of X. Then there is an expanding sequence $(F_n)_{n\in\mathbb{N}}$ of functionally closed subset of X such that $U = \bigcup_{n\in\mathbb{N}} F_n$.*

P r o o f. Let $f\colon X \to [0,1]$ be a continuous function such that $U = f^{-1}(0,1]$. Put $F_n = f^{-1}(\frac{1}{n}, 1]$ for every $n \in \mathbb{N}$. It is easy to verify that each set F_n is functionally closed and $U = \bigcup_{n\in\mathbb{N}} F_n$. □

Lemma 3.5. *Let U and V be functionally closed subset of X. Then there is a continuous function $f\colon X \to [0,1]$ such that $f^{-1}(0) = U$, $f^{-1}(1) = V$.*

Proof. See [2], page 43.

Lemma 3.6. *Let S and T be nonempty sets and let $\{X_{s,t}\}_{(s,t)\in S\times T}$ be a family of subsets of X. Then*

$$\bigcup_{s\in S}\bigcap_{t\in T} X_{s,t} = \bigcap_{f\in T^S}\bigcup_{s\in S} X_{s,f(s)}.$$

Proof. Put $A = \bigcup_{s\in S}\bigcap_{t\in T} X_{s,t}$, $B = \bigcap_{f\in T^S}\bigcup_{s\in S} X_{s,f(s)}$.

Let $x \in A$. Then there is $s_0 \in S$ such that $x \in X_{s_0,t}$ for all $t \in T$. Let $f \in T^S$. Then $x \in X_{s_0,f(s_0)}$, hence $x \in \bigcup_{s\in S} X_{s,f(s)}$, which implies that $x \in B$, i.e., that $A \subset B$.

Let $x \notin A$. Then for each $s \in S$ there is $t = f(s) \in T$ such that $x \notin X_{s,f(s)}$; hence $x \notin \bigcup_{s\in S} X_{s,f(s)}$ and $x \notin B$, i.e., $B \subset A$. $\square$

The previous lemma implies the following corollary.

Corollary 3.7. *If in the condition of the previous lemma we require that S and T should be countable and each set $X_{s,t}$ should be open in X, then the set $\bigcup_{s\in S}\bigcap_{t\in T} X_{s,t}$ is a G_c-subset of X.*

Lemma 3.8. *The set $B_s = \{x \in X : K(x) \cap (V \setminus V_s) \neq \varnothing\}$ is a G_δ-subset of X.*

Proof. Let $(F_n^s)_{n\in\mathbb{N}}$ be a decomposition of V_s satisfying the following conditions:

$$F_n^s \in \mathcal{C} \text{ and } F_n^s \subset F_{n+1}^s \text{ for all } n \in \mathbb{N}.$$

Put $U_n = V \setminus F_n^s$. Then $V \setminus V_s = \bigcap_{n\in\mathbb{N}} U_n$, where each U_n is open and $U_n \supset U_{n+1}$ for all $n \in \mathbb{N}$. Let $C_s = \bigcap_{n\in\mathbb{N}} K^{-1}(U_n)$. We shall show that $B_s = C_s$. The inclusion $B_s \subset C_s$ is obvious. Let $x \in C_s$. Since $K(x)$ is finite, there is $y \in K(x)$ such that $y \in U_n$ for all n in some infinite subset of $\mathbb{N}$. Hence, $y \in \bigcap_{n\in\mathbb{N}} U_n$ and $x \in B_s$. By Corollary 1.5 on page 147, the set $K^{-1}(U_n)$ is a G_δ-subset of X for all $n \in \mathbb{N}$. This implies that B_s, as a countable intersection of G_δ-sets, is a G_δ-set. $\square$

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